Math 215 - Problem Set 8: Green's Theorem, Curl and Divergence, Parametric Surfaces and thier Areas

Math 215 SI

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1 Review

1.1 Green's Theorem

Theorem 1 (Green's Theorem)**.** *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ are functions of (x,y) with continuous partial derivatives on an open region that contains D, then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

In this theorem, P and Q are the components of a vector field $\mathbf{F} = (P, Q)$, and the left-hand side is the line integral of $\mathbf{F}$ around the curve C. The right-hand side is the double integral of the curl of $\mathbf{F}$ over the region D.

1.2 Curl and Divergence

1.2.1 Curl

Theorem 2 (Curl)**.** *Let $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field in $\mathbb{R}^3$. The curl of $\mathbf{F}$ is defined as*

$$curl\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

The curl of a vector field measures the rotation or "twisting" of the field at a point. It is a vector quantity that points in the direction of the axis of rotation, and its magnitude represents the strength of the rotation.

Conservative Fields. If the curl of a vector field $\mathbf{F}$ is zero, i.e., $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is said to be a conservative vector field. This means that $\mathbf{F}$ can be expressed as the gradient of a scalar potential function ϕ, such that $\mathbf{F} = \nabla\phi$. In physical terms, this implies that the work done by the field along any closed path is zero, and the field is path-independent.

1.2.2 Divergence

Theorem 3 (Divergence)**.** *Let $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field in $\mathbb{R}^3$. The divergence of $\mathbf{F}$ is defined as*

$$div\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a vector field measures the "spreading out" or "convergence" of the field at a point. It is a scalar quantity that indicates whether the field is diverging from or converging towards that point.

Propisition. The divergence of the curl of any vector field is zero, i.e., $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. This means that the curl of a vector field has no net "outflow" at any point in space. This is a consequence of the fact that the curl measures rotation, while divergence measures "spreading out".

1.3 Parametric Surfaces and their Areas

Definition 1. *A parametric surface is a surface in $\mathbb{R}^3$ defined by a vector-valued function $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, where (u, v) are parameters that vary over some region in the uv-plane. The surface is the image of this parameterization.*

Consider the vector function $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$

- The partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v are given by:

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right), \quad \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

  These vectors lie in the tangent plane to the surface at a given point.

- The normal vector to the tangent plane at a point on the surface is given by the cross product of the partial derivatives:

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}.$$

  This vector is perpendicular to the tangent plane and can be used to compute the surface area.

2 Problems

2.1 Problem 1

$$\mathbf{F} = < xyz, y\sin(z), y\cos(x) >$$

- Compute the curl of $\mathbf{F}$.
- Compute the divergence of $\mathbf{F}$.

2.2 Problem 2

Suppose that D is the bounded region in the plane that has boundary given by the oriented simple closed piecewise smooth curves $C_1, C_2, C_3, \text{and}, C_4$ as in the picture. Suppose $\mathbf{F} =< P, Q, 0 >: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and P and Q hav continuous partial derivaties on $\mathbb{R}^2$. If

$$\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}$$

find $\int\int_D (Q_x - P_y) dA = \int\int_D (\nabla \times F) \cdot \hat{k} dA$.

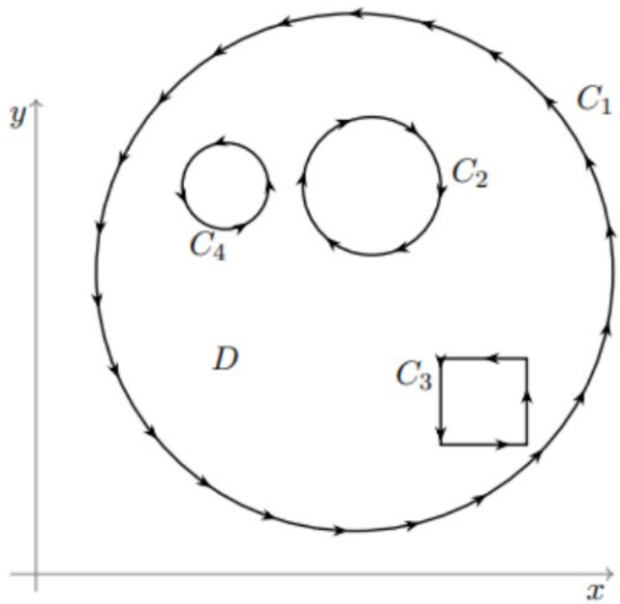

Figure 1: Boundary curves C_1, C_2, C_3, C_4 of the region D.

2.3 Problem 3

Evaluate the line integral $\int_C y^3 dx + (3x - x^3) dy$, where C is the curve $x^2 + y^2 = 1$ oriented counterclockwise, using Green's Theorm.