Math 215 - Problem Set 9: Surface Integrals (Scalar and Vector Fields), Stokes' Theorem

Math 215 SI

April 12, 2025

1 Review

1.1 Surface Integrals (Scalar Functions)

Definition 1. *Given a surface S parameterized by $\mathbf{r}(u,v)$, the surface integral of a scalar function $f(x,y,z)$ over the surface S is defined as:*

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv$$

where D is the parameter domain in the uv-plane, and $\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|$ is the magnitude of the cross product of the partial derivatives, which gives the area element of the surface.

Note that to compute the area of the surface integral S, we can set $f(x,y,z) = 1$. The surface integral then becomes the area of the surface:

$$\iint_S dS = \iint_D \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv$$

1.2 Surface Integrals (Vector Fields)

Definition 2. *Given a surface S parameterized by $\mathbf{r}(u,v)$ with a unit normal vector $\mathbf{n}$, the surface integral of a vector field $\mathbf{F}(x,y,z)$ over the surface S is defined as:*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv$$

where D is the parameter domain in the uv-plane, and $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is the cross product of the partial derivatives, which gives the oriented area element of the surface.

- If $\mathbf{F}$ is a velocity field of a fluid, then the flux

  $$\int \int_S \mathbf{F} \cdot \mathbf{n}\,dS$$

  is the rate of flow across the surface S.

- The orientation of S is determined by the choice of the unit normal vector $\mathbf{n}$. The direction of $\mathbf{n}$ can be chosen to be outward or inward, depending on the context of the problem.

1.3 Stokes' Theorem

Theorem 1 (Stokes' Theorem)**.** *Let S be a smooth, oriented surface with boundary curve C, and let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. Then,*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $d\mathbf{S} = \mathbf{n}\, dS$ is the oriented area element of the surface S, and $d\mathbf{r}$ is the line element along the curve C.

If $S = D$ is a domain in $\mathbb{R}^2$ and $C = \partial D$ is the boundary of D, then Stokes' theorem reduces to Green's theorem.

2 Problems

2.1 Problem 1

Consider the vector field $\mathbf{F} = (3x + 2yz, 2x - y + z, x - 3y + 2z)$ and the unit cube in the first octant. What is the flux upwards through the top surface of the cube?

2.2 Problem 2

Suppose R is a positive real number. Let S be the cone given by the equation $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq R$, oriented downward. Compute the flux of $G = (xz, yz, xy)$ across S.

2.3 Problem 3

Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = 3z\mathbf{i} + (x + \frac{z^2}{2})\mathbf{j} + (2y + yz)\mathbf{k}$$

(a) Evaluate the curl $\nabla \times \mathbf{F}$.

(b) Evaluate the line integral (Circulation)

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

with $\mathbf{r}$ being the position vector of a point on the closed curve C (as usual) and C is the triangle joining the points

$$A = (2, 1, -1), B = (1, 2, -2), C = (4, 0, -2).$$

The orientation of C is from A to B to C and back to A

2.4 Problem 4

Consider the paraboloid surface P given by $z = 1 - (x^2 + y^2)$, $0 \leq x^2 + y^2 \leq 1$.

(a) Find the area of the surface P.

(b) Evaluate the surface integral

$$\int\int_P \mathbf{F} \cdot \mathbf{n} \, dS$$

where P is the same paraboloid surface and $\mathbf{F} = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j} + z\mathbf{k}$ and n is the unit normal vector to the surface P (in the upward direction) and dS is the area element. The disc at the bottom is not included in the surface.