Math 215 – Midterm II Review

Math 215 SI
March 2026

1. Maximum and Minimum Values (14.7)

Critical Points: Find where $\nabla f = \langle f_x, f_y \rangle = 0$.
Second Derivative Test:

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

- $D > 0$, $f_{xx} > 0$: local min
- $D > 0$, $f_{xx} < 0$: local max
- $D < 0$: saddle point
- $D = 0$: inconclusive

2. Constrained Optimization & Lagrange Multipliers (14.8)

$$\nabla f = \lambda \nabla g$$

3. Double Integrals (15.1–15.4)

Rectangular Regions

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

General Regions

Type I:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II:

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Polar Coordinates

$$\iint_R f(x,y)\,dA = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

4. Applications of Double Integrals (15.5–15.6)

Area:

$$\iint_R 1\,dA$$

Volume:

$$\iint_R f(x,y)\,dA$$

Average Value:

$$f_{\text{avg}} = \frac{1}{\text{Area}}\iint_R f(x,y)\,dA$$

Surface Area:

$$\iint_R \sqrt{1+f_x^2+f_y^2}\,dA$$

5. Triple Integrals (15.7–15.9)

Cartesian Coordinates

$$\iiint_E f(x,y,z)\,dV$$

Cylindrical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$dV = r\,dz\,dr\,d\theta$$

Spherical Coordinates

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$
$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

6. Change of Variables

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dA$$

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Problems

0.1 Problem - on Maximum and Minimum Values

Consider the function $f(x, y) = x(9 - x^2 - y^2)$ defined for all of $\mathbb{R}^2$

(a) Find all critical points of f, and classify each as a local minimum, local maximum, or saddle

(b) On the domain D for which $x \leq 0$ and $x^2+y^2 \leq 16$, find the absolute maximum and minimum values of f, and the points at which they occur.

0.2 Problem - Lagrange Multipliers

Spider-Man decides to annoy J. Jonah Jameson by swinging by the Daily Bugle and throwing a pie at Jameson's office window. For maximum hilarity, it is important that Spider-Man be sufficiently close to the window that the pie makes a startling and satisfying "splatt!" noise, but not so close that the force of the thrown pie cracks the window.
Suppose that the entrance to the Daily Bugle is located at $(0, 0, 0)$, and Jameson's office is located at the point $(2, 0, 14)$. Further suppose that the arc of Spider-Man's swing is constrained to lie on the two-sheeted hyperboloid

$$x^2 - y^2 + z^2 = -50, \quad y \geq 0.$$

Find the point on Spider-Man's trajectory that is closest to Jameson's office.

0.3 Problem - Triple Integrals and Projections

Consider the triple integral given by

$$\iiint_E f(x, y, z)\, dV = \int_{z=0}^{1} \int_{y=z}^{1} \int_{x=z}^{1} f(x, y, z)\, dx\, dy\, dz.$$

(a) Sketch the volume E relative to the (x, y, z) coordinate system axes.

(b) Sketch the projection of E (i.e. the shadow that E casts) into each of the coordinate planes: (x, y), (x, z), and (y, z).

(c) Write the integral such that the order of integration is $dzdydx$

0.4 Problem - Change of Variables

Let R be the region in the first quadrant bounded by the curves

$$x^2 - y^2 = 4, \quad x^2 - y^2 = 16, \quad x + y = 4, \quad \text{and} \quad x + y = 8.$$

There is a coordinate transformation $(x, y) \mapsto (u, v)$ that maps the region R in the xy-plane into a rectangular region D in the uv-plane.

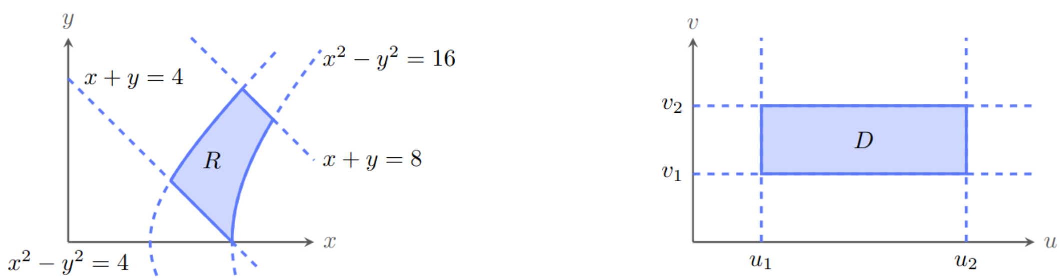

Figure 1: Caption describing the figure

(a) (3 points) Find $u(x, y)$ and $v(x, y)$ that transform R into D.

(b) (2 points) Find the values for u_1, u_2, v_1, and v_2.

(c) (4 points) Recall the Jacobian of a transformation $(u, v) \mapsto (x, y)$ is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

If it helps, you may take as given that

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1}.$$

Compute the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.

(d) Set up but don't solve $\iint_R 2(x - y)dA$ in in terms of u and v