Math 215 – Problem Set 2

Vector Functions and Space Curves, Limits, Derivatives and Integrals of Vector Functions, Arc Length, Motion in space.

Math 215 SI
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2 Review

2.1 Vector Functions and Space Curves

A vector function is a mapping from some domain (typically an interval of real numbers) to a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. It can be written in component form as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

A space curve is the geometric object traced out by a vector function as the parameter t varies. The parameter t often represents time, but it may also represent arc length or another physical quantity.

2.2 Derivatives and Integrals of Vector Functions

The derivative of a vector function is defined component-wise:

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Geometrically, $\mathbf{r}'(t)$ represents the tangent vector to the curve at time t.
Integrals of vector functions are also computed component-wise:

$$\int \mathbf{r}(t)\, dt = \left\langle \int x(t)\, dt,\ \int y(t)\, dt,\ \int z(t)\, dt \right\rangle + \mathbf{C}.$$

Vector integrals are commonly used to recover position from velocity or velocity from acceleration.

2.3 Arc Length

The arc length of a smooth space curve $\mathbf{r}(t)$ on the interval $[a, b]$ is given by

$$L = \int_a^b \|\mathbf{r}'(t)\|\, dt,$$

where $\|\mathbf{r}'(t)\|$ is the magnitude of the velocity vector. Arc length measures the total distance traveled along the curve.

2.4 Motion in Space

When modeling motion in space, the vector function $\mathbf{r}(t)$ represents position, $\mathbf{v}(t) = \mathbf{r}'(t)$ represents velocity, and $\mathbf{a}(t) = \mathbf{v}'(t)$ represents acceleration. The speed of the particle is the magnitude of the velocity:

$$\text{speed} = \|\mathbf{v}(t)\|.$$

Velocity can be decomposed into tangential and normal components, which describe how the speed and direction of motion change over time.

2 Problems

Problem 1

(a) Find the arc length of the curve $r(t) =< cos(t), sin(t), \frac{2}{3}t^{\frac{3}{2}}$ for $0 \leq t \leq \pi$

(b) Find the equation of the tangent line to the space curve at $t = 0$

(c) If t is time find the speed of the particle at $t = 2\pi$

Problem 2

In this problem all coordinates are measured in meters and time is measured in seconds. At time $t = 0$ a ladybug, named Sam, is at position $(1, 1, 1)$ and is flying with constant velocity $< 1, 2, 3 >$ meters per second. A sensor placed at $(3, 6, 7)$ can detect ladybug motion that occurs within a sphere of radius 7 (meters). Does the sensor detect Sam? If so at what time is Sam last detected by the sensor.

Problem 3

The trajectory of a particle is given by $r(t) =< \sqrt{3}t^2, 2t^2, \sqrt{6}t^2 >$. Let C denote the corresponding space curve.

(a) Find an equation for the tangent line to C at the point $(4\sqrt{3}, 16, 4\sqrt{6})$.

(b) How long is C on the interval $0 \leq t \leq \sqrt{8}$