# Math 215 – Problem Set 5

Double Integrals and Polar Coordinates

Math 215 SI
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## Review

### 1. Double Integrals Over Rectangles

For a function $f(x, y)$ defined on a rectangle

$$R = [a, b] \times [c, d]$$

the double integral is defined as the limit of Riemann sums:

$$\iint_R f(x, y)\, dA$$

Using Fubini's Theorem, we can compute the double integral as an iterated integral:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

or

$$\int_c^d \int_a^b f(x, y)\, dx\, dy$$

depending on which order is easier.

### 2. Double Integrals Over General Regions

Let $f(x, y)$ be continuous over a region $D$.

**Type I Region**

A region is Type I if it can be described as

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

Then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

**Type II Region**

A region is Type II if it can be described as

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

Then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

## 3. Useful Properties

### Area of a Region

$$\text{Area}(D) = \iint_D 1\, dA$$

### Average Value of a Function

$$f_{\text{avg}} = \frac{1}{\text{Area}(D)} \iint_D f(x, y)\, dA$$

**Symmetry**

If $D$ is symmetric about the $x$-axis and $f(x, y)$ is odd in $y$, then

$$\iint_D f(x, y)\, dA = 0$$

Similarly, symmetry about the $y$-axis can simplify integrals if $f$ is odd in $x$.

## 4. Double Integrals in Polar Coordinates

Polar coordinates are useful for circular regions.

$$x = r\cos\theta, \quad y = r\sin\theta$$

The area element becomes

$$dA = r\, dr\, d\theta$$

Thus

$$\iint_D f(x, y)\, dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

# Problems

## Problem 1

Consider the double integral

$$\iint_D xy\,dA$$

over the triangular region $D$ bounded by the three lines

$$x = 0, \quad y = 0, \quad x + y = 1$$

(i) What are the three vertices of $D$?

(ii) Evaluate the integral.

## Problem 2

Compute the following integral

$$\int_0^{1/2} \int_{y^3}^{1-y^2} 30xy^2 \, dx \, dy$$

## Problem 3

Consider the function

$$f(x,y) = x - y$$

on the region

$$D = \{(x,y) \mid x^2 + y^2 \leq 4,\ x + y \geq 0\}$$

(i) Sketch the region $D$.

(ii) Calculate the integral of $f$ over $D$.

## Problem 4

Let $R$ be the region inside the circle

$$x^2 + (y-1)^2 = 1$$

but outside the circle

$$x^2 + y^2 = 1$$

(i) Find the bounds of the region $R$ in polar coordinates.

(ii) Suppose we have a lamina in the shape of $R$ with mass density

$$f(x, y) = \frac{1}{x^2 + y^2} - \frac{x}{x^2 + y^2}$$

Find the total mass $M$ of the lamina.