Math 215 – Problem Set 7

Vector Fields, Line Integrals, and the Fundamental Theorem for Line Integrals

Math 215 SI
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5 Review

5.1 Vector Fields

A **vector field** is a function $\vec{F}$

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

Examples by dimension:

- $n = 2$: $\vec{F}(x, y) = \big(P(x, y),\, Q(x, y)\big)$
- $n = 3$: $\vec{F}(x, y, z) = \big(P(x, y),\, Q(x, y),\, R(x, y, z)\big)$

Key properties:

- $\vec{F}$ is continuous $\iff$ its coordinate functions are continuous.
- $\vec{F}$ is differentiable $\iff$ its coordinate functions are differentiable.

5.2 Line Integrals

Let C be a curve parameterized by $\vec{r}(t)$ on $a \le t \le b$.

(1) Scalar line integral: Given a scalar function f, its line integral along C is

$$\int_C f\, ds := \int_a^b f\big(\vec{r}(t)\big)\,\big|\vec{r}\,'(t)\big|\, dt$$

(2) Vector line integral: Given a vector field $\vec{F}$, its line integral along C is

$$\int_C \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt$$

If $\vec{F}$ is a force vector field, then $\int_C \vec{F} \cdot d\vec{r}$ is the work done by $\vec{F}$ along C.

5.3 Fundamental Theorem for Line Integrals

Theorem: Let C be a curve parameterized by $\vec{r}(t)$ on $a \le t \le b$. If $\vec{F}$ is a **conservative** vector field with potential function f, then

$$\int_C \vec{F} \cdot d\vec{r} = f\big(\vec{r}(b)\big) - f\big(\vec{r}(a)\big)$$

where $\vec{r}(b)$ is the terminal value and $\vec{r}(a)$ is the initial value.

This says the work done by $\vec{F}$ is given by the difference in potential. In particular, if C is a **closed curve**,

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

When is $\vec{F}$ conservative? If $\vec{F} = (P, Q)$ is conservative with potential function f, then $P = f_x$ and $Q = f_y$, so

$$\frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x}$$

on D, where D is an open and simply connected domain in $\mathbb{R}^2$.

Problems

Problem 1

Consider the force field $\vec{F}(x, y) = (2x - y,\ 4y - x)$.

(i) Is $\vec{F}$ conservative on $\mathbb{R}^2$?

(ii) Find a potential function f of $\vec{F}$.

(iii) Find the work done by $\vec{F}$ along a curve C_1 from $(1, 0)$ to $(2, 1)$.

(iv) Find a curve C_2 with

$$\int_{C_2} \vec{F} \cdot d\vec{r} = 2$$

Problem 2

Consider the vortex field

$$\vec{V}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

(i) Is $\vec{V}$ conservative on the domain $y > 0$?

(ii) Find $\int_C \vec{V} \cdot d\vec{r}$ where C is given by $x^2 + (y-2)^2 = 1$.

Problem 3

Find the work done by the force field

$$\vec{F}(x, y, z) = (x + y,\ y^2 - z,\ 2z)$$

along the line segment C from $(0, 0, 1)$ to $(2, 1, 0)$.