Math 215 – Problem Set 8

Green's theorem, curl and divergence, parametric surfaces and surface area.

Math 215 SI
April 5, 2026

8 Review

8.1 Green's theorem

Let D be a domain in $\mathbb{R}^2$ with boundary ∂D. The boundary is *simple* if it has no self-intersections. It is *positively oriented* if, as you traverse each component, the interior of D lies on your left.

If $\mathbf{F} = \langle P, Q\rangle$ is differentiable on D and ∂D is simple and positively oriented, then

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

For $\mathbf{F} = \langle P, Q, 0\rangle$ this equals $\displaystyle\iint_D (\nabla \times \mathbf{F})\cdot \mathbf{k}\, dA$.

8.2 Curl and divergence

Write $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$. For $\mathbf{F} = \langle P, Q, R\rangle$,

$$\operatorname{curl}(\mathbf{F}) = \nabla\times\mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle, \qquad \operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Curl is a vector field; divergence is a scalar. On a plane domain D with simple positively oriented boundary, $\displaystyle\int_{\partial D}\mathbf{F}\cdot d\mathbf{r} = \iint_D \operatorname{curl}(\mathbf{F})\cdot\mathbf{e}_3\, dA$ for differentiable $\mathbf{F} = \langle P, Q, 0\rangle$.

If $\mathbf{F}$ is conservative if and only if $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$ and $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$.

8.3 Parametric surfaces and area

A parametric surface is traced by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. The graph $z = f(x, y)$ is parametrized by $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$. Partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ span the tangent plane; a normal vector is $\mathbf{r}_u \times \mathbf{r}_v$. The surface area element is $\|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$.

8 Problems

Problem 1

[8 points] As usual, justify your answers.

a. [4 points] Suppose that D is the bounded region in the plane whose boundary consists of the oriented simple closed piecewise smooth curves C_1, C_2, C_3, and C_4 as in the picture. Suppose $\mathbf{F} = \langle P, Q, 0\rangle : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and P and Q have continuous partial derivatives on $\mathbb{R}^2$. If

$$\oint_{C_k} \mathbf{F} \cdot d\mathbf{r} = 2^k,$$

find $\iint_D (Q_x - P_y)\, dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$.

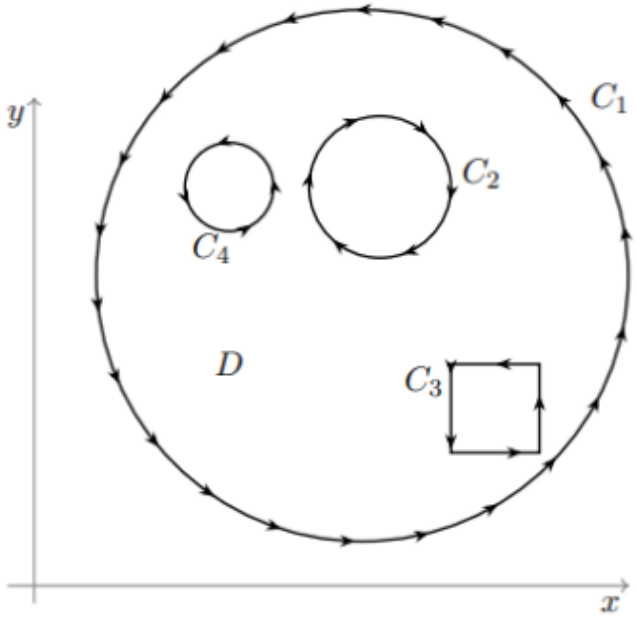

Figure 1: Problem 1

Problem 2

Let $\mathbf{F} = \langle xyz,\, y\sin(z),\, y\cos(x)\rangle$.

(i) Compute curl($\mathbf{F}$).

(ii) Compute div($\mathbf{F}$).

Problem 3

Evaluate the line integral $\int_C y^3\,dx+(3x-x^3)\,dy$, where C is the curve $x^2+y^2=1$ (counterclockwise sense), using Green's theorem.