Math 215 – Problem Set 9

Surface integrals (scalar functions and vector fields), Stokes' theorem.

Math 215 SI
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9 Review

9.1 Surface integrals of a scalar function

If S is parametrized by $\mathbf{r}(u,v)$ on a domain D in the uv-plane, the surface integral of a scalar function f over S is

$$\iint_S f\,dS := \iint_D f(\mathbf{r}(u,v))\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA,$$

where $\|\mathbf{r}_u \times \mathbf{r}_v\|$ plays the role of a Jacobian factor. In particular,

$$\text{Area}(S) = \iint_S 1\,dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

9.2 Surface integrals of vector fields (flux)

With the same parametrization, the flux of a vector field $\mathbf{F}$ across S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} := \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA.$$

If $\mathbf{F}$ is a fluid velocity, this integral is the rate of flow across S. The orientation of S is fixed by the choice of normal (e.g. replacing $\mathbf{r}_u \times \mathbf{r}_v$ by its negative reverses orientation).

9.3 Stokes' theorem

The boundary curve ∂S of an oriented surface S is *positively oriented* if, when walking along ∂S in the direction of its orientation, the interior of S lies on your left (consistent with the chosen normal via the right-hand rule).

If $\mathbf{F}$ is differentiable on S and ∂S is simple and positively oriented, then

$$\int_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F})\cdot d\mathbf{S}.$$

When S is a flat region $D \subset \mathbb{R}^2$ (with standard upward normal), this reduces to Green's theorem in the plane. Stokes' theorem is useful for converting between circulation line integrals and flux integrals of $\text{curl}(\mathbf{F})$.

9 Problems

Problem 1

Consider the vector field $\mathbf{F} = \langle 3x + 2yz, 2x - y + z, x - 3y + 2z\rangle$ and the unit cube in the first octant. What is the flux upward through the top surface of the cube?

Problem 2

Suppose R is a positive real number. Let S be the cone given by $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq R$, oriented downward. Compute the flux of $\mathbf{G} = \langle xz, yz, xy \rangle$ across S.

Problem 3

8. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = 3z\,\mathbf{i} + \left(x + \frac{z^2}{2}\right)\mathbf{j} + (2y + yz)\,\mathbf{k}.$$

(a) (2 points) Evaluate the curl $\nabla \times \mathbf{F}$.

(b) (8 points) Evaluate the line integral (circulation)

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

with $\mathbf{r}$ the position vector on the closed curve C, where C is the triangle joining the points

$$A = (2, 1, -1), \quad B = (1, 2, -2), \quad C = (4, 0, -2).$$

The orientation of C is from A to B, B to C, and then C back to A.

Problem 4

6. Consider the paraboloid surface P given by $z = 1 - (x^2 + y^2)$, $0 \leq x^2 + y^2 \leq 1$.

(a) (5 points) Find the area of the surface P.

(b) (5 points) Evaluate the surface integral

$$\iint_P \mathbf{F} \cdot \mathbf{n} \, dS,$$

where P is the same paraboloid surface and $\mathbf{F} = \frac{x}{2}\,\mathbf{i} + \frac{y}{2}\,\mathbf{j} + z\,\mathbf{k}$. As usual, $\mathbf{n}$ is the unit normal to the surface in the upward direction and dS is the area element. The disc at the bottom is *not* included in the surface.