# MATH 215 SI

Session 1

# Session 1

## Lectures 1 - 4

- 12.1 Three-Dimensional Coordinate Systems
- 12.2 Vectors
- 12.3 The Dot Product
- 12.4 The Cross Product
- 12.5 Equations of Lines and Planes

# 12.1 Three-Dimensional Coordinate Systems

- Defined as $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
- We represent any point of space by an ordered triple $(a, b, c)$

# Exercises

## 12.1 Three-Dimensional Coordinate Systems

- What does the equation $x = 4$ represent in $\mathbb{R}^2$?
- What does the same equation represent in $\mathbb{R}^3$?
- Describe and/or sketch the surface in $\mathbb{R}^3$ represented by the equation $x + z = 2$

# 12.2 Vectors

We will define a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

- Then the magnitude of $\mathbf{v}$ is given by $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- Then the unit vector (direction) of $\mathbf{v}$ is given by $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$
- Vector addition is given by $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, ..., v_n + u_n \rangle$
- Scalar multiplication by $c$ is given by $c\mathbf{v} = \langle cv_1, cv_2, ..., cv_n \rangle$
- The standard basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors that point in the direction of the positive $x$-, $y$-, and $z$-axes.
- Recall the Parallelogram Law

# 12.2 Vectors

cont.

Properties of Vectors

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
- $1\mathbf{a} = \mathbf{a}$

# Exercises

## 12.2 Vectors

- Generate an expression that is equivalent to the vector $\langle 11, 3, 5\rangle$ as a linear combination of $\mathbf{v} = \langle 5, 1, 2\rangle$ and the standard basis vectors (each vector should appear in your expression).
- Find a vector that has the same direction as $\langle 6, 2, -3\rangle$ but has length 4.

# 12.3 The Dot Product

If $\mathbf{a} = \langle a_1, a_2, ..., a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, ..., b_n \rangle$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

- The dot product measures how aligned two vectors are
- If $\theta$ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal $\iff \mathbf{a} \cdot \mathbf{b} = 0$

# 12.3 The Dot Product

cont.

Properties of the Dot Product

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
- $\mathbf{0} \cdot \mathbf{a} = 0$

# 12.3 The Dot Product

## cont.

Projections

- The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by $\text{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$
- The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by $\text{proj}_a \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|}$

# Exercises

## 12.3 The Dot Product

- Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and let $\theta_1$ be the angle $\mathbf{v}$ makes with the $x$-axis, $\theta_2$ be the angle $\mathbf{v}$ makes with the $y$-axis, and $\theta_3$ be the angle $\mathbf{v}$ makes with the $z$-axis. Evaluate the following expression: $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3$

# 12.4 The Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b}$ given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta =$ area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$
- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel $\iff$ $\mathbf{a} \times \mathbf{b} = 0$

# 12.4 The Cross Product

cont.

Properties of the Cross Product

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

# Exercises

## 12.4 The Cross Product

- Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

# 12.5 Equations of Lines and Planes

A line through point $\mathbf{r}_0$ with direction $\mathbf{v}$ is given by $\mathbf{r}(t) = t\mathbf{v} + \mathbf{r}_0$

- Note that $\mathbf{r}$ is a function of $t$
- By eliminating $t$ we can yield the symmetric equations (not useful in my experience).

A plane through point $\mathbf{r}_0$ with normal vector $\mathbf{n}$ is given by $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

- Note that $\mathbf{r}$ is an arbitrary vector
- Expanding the expression we yield the scalar equation of the plane given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

# 12.5 Equations of Lines and Planes

cont.

Distances

- The distance $D$ from point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

# Exercises

## 12.5 Equations of Lines and Planes

Suppose $P_1$ is a plane that contains the points $(0,3,0)$, $(-3,0,0)$, and $(2,6,1)$.

- Find the equation of the plane $P_1$

Let $P_2$ be the plane that contains the point $(0,2,1)$ and the line $\mathbf{l}(t) = \langle 2t, t, 1+3t \rangle$

- Find the angle between $P_1$ and $P_2$

# Fin

This concludes session 1.