MATH 215 SI

Session 5

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Lectures 18 - 22

- 14.8 Lagrange Multipliers (revisited)
- 15.1 Double Integrals over Rectangles
- 15.2 Double Integrals over General Regions
- 15.3 Double Integrals in Polar Coordinates

14.8 Lagrange Multipliers (revisited)

To find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming they exist and $\nabla g \neq \mathbf{0}$ on the constraint):

1. Find all x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad g(x, y, z) = k$$

2. Evaluate f at all the points found in step 1. The largest is the maximum, the smallest is the minimum

- Geometrically, at an extremum the level curve/surface of f is tangent to the constraint, so $\nabla f \parallel \nabla g$

14.8 Lagrange Multipliers (revisited)

cont.

Two Constraints

- To find extreme values of $f(x, y, z)$ subject to $g(x, y, z) = k$ and $h(x, y, z) = c$, solve

$$\nabla f = \lambda \nabla g + \mu \nabla h \qquad g(x, y, z) = k \qquad h(x, y, z) = c$$

Combining with 14.7

- To find absolute extrema on a closed region, use Lagrange multipliers to handle the boundary, and compare with the critical points in the interior

Exercises

14.8 Lagrange Multipliers

- Find the maximum and minimum values of the function $f(x, y) = x + y$ on the curve $x^2 + y^2 - xy = 4$
- Consider a hypothetical planet whose surface can be approximated by the sphere $x^2 + y^2 + z^2 = 1$. Suppose the planet's temperature is strange and given by $T(x, y, z) = 2x + 2y + z$. Find the maximum and minimum temperatures on the planet's surface

15.1 Double Integrals over Rectangles

Let $R = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$. The double integral of f over R is

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

- If $f(x, y) \geq 0$, then $\iint_R f(x, y)\, dA$ is the volume of the solid above R and below the surface $z = f(x, y)$
- The average value of f over R is $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$

15.1 Double Integrals over Rectangles

cont.

Fubini's Theorem

If f is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

- Work from the inside out: integrate the inner variable while treating the other as a constant
- The order does not change the answer, but it can make the integral much easier (or harder)
- If $f(x, y) = g(x)h(y)$, then $\iint_R g(x)h(y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$

Exercises

15.1 Double Integrals over Rectangles

- Evaluate $\iint_R xy^2\,dA$ where $R = [0,2] \times [1,3]$
- Evaluate $\iint_R ye^{xy}\,dA$ where $R = [0,2] \times [0,1]$. Which order of integration is easier?

15.2 Double Integrals over General Regions

Type I: $D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Type II: $D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

- Always sketch the region first. The outer limits must be constants, the inner limits can depend on the outer variable

15.2 Double Integrals over General Regions

cont.

Properties

- $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$ and $\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA$
- If $D = D_1 \cup D_2$ (not overlapping except on boundaries), then $\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$
- $\iint_D 1\, dA = A(D)$, the area of D
- If $m \le f(x,y) \le M$ on D, then $m\, A(D) \le \iint_D f(x,y)\, dA \le M\, A(D)$

Exercises

15.2 Double Integrals over General Regions

- Let D be the triangular region bounded by $x = 0$, $y = 0$, and $x + y = 1$
  - What are the three vertices of D?
  - Evaluate $\iint_D xy\,dA$
- Compute $\displaystyle\int_0^{1/2} \int_{y\sqrt{3}}^{\sqrt{1-y^2}} 30xy^2\,dx\,dy$

15.3 Double Integrals in Polar Coordinates

Recall $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$

If f is continuous on the polar rectangle $R = \{(r,\theta) \mid 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$ where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

- Don't forget the extra factor of r: $dA = r\,dr\,d\theta$
- Polar is usually the right choice when the region is a disk, annulus, or sector, or when f involves $x^2 + y^2$

15.3 Double Integrals in Polar Coordinates

cont.

General Polar Regions

If $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$, then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Exercises

15.3 Double Integrals in Polar Coordinates

- Consider $f(x, y) = x - y$ on the region D defined by $x + y \geq 0$ and $x^2 + y^2 \leq 4$
  - Sketch the region D
  - Calculate the integral of f over D

Exercises

15.3 Double Integrals in Polar Coordinates (cont.)

- Let R be the region inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 1$
  - Find the bounds of R in polar coordinates
  - A lamina in the shape of R has mass density $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} - \dfrac{x}{x^2+y^2}$. Find the total mass M of the lamina

Fin

This concludes session 5.