MATH 215 SI

Session 6

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Lectures 23 - 25

- 15.4 Applications of Double Integrals
- 15.5 Surface Area
- 15.6 Triple Integrals
- 15.7 Triple Integrals in Cylindrical Coordinates

15.4 Applications of Double Integrals

Suppose a lamina occupies a region D in the xy-plane and has density $\rho(x, y)$ (mass per unit area)

- The total mass is

$$m = \iint_D \rho(x, y)\, dA$$

15.4 Applications of Double Integrals

cont.

Moments and Center of Mass

- The moment about the x-axis and the moment about the y-axis are

$$M_x = \iint_D y\,\rho(x,y)\,dA \qquad M_y = \iint_D x\,\rho(x,y)\,dA$$

- The center of mass $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

Exercises

15.4 Applications of Double Integrals

- Consider the half-disc $0 \leq x^2 + y^2 \leq 1$ with $y \geq 0$. Assume that the density is $\rho(x, y) = 1$
  - Find $\bar{y}$, the y-coordinate of the center of mass of the half-disc
  - Find the y-coordinate of the center of mass of the triangular region with vertices $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, and $(0, \frac{1}{2})$, with the same density

15.5 Surface Area

The area of the surface $z = f(x, y)$ over a region D, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$$

- Equivalently, $A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$
- Compare with arc length $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Exercises

15.5 Surface Area

- Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $A(D)\sqrt{a^2 + b^2 + 1}$.

15.6 Triple Integrals

Fubini's Theorem for a box $B = [a, b] \times [c, d] \times [r, s]$:

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

- Any of the six orders of integration gives the same answer
- $\iiint_E 1\, dV = V(E)$, the volume of E
- Mass and center of mass work just like 15.4: $m = \iiint_E \rho\, dV$, $\bar{x} = \frac{1}{m} \iiint_E x\rho\, dV$, etc.

Exercises

15.6 Triple Integrals

- Evaluate $\iiint_B xyz\,dV$ for the following choices of B
  - B is the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$
  - B is the region $0 \leq x \leq y \leq z \leq 1$

15.7 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates are polar coordinates in the xy-plane with z left alone:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad r^2 = x^2 + y^2$$

If $E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$ and D is given in polar by $\alpha \le \theta \le \beta$, $h_1(\theta) \le r \le h_2(\theta)$, then

$$\iiint_E f\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

- Again, don't forget the r: $dV = r\,dz\,dr\,d\theta$
- Good choice when E has symmetry about the z-axis (cylinders, cones, paraboloids) or f involves $x^2 + y^2$

Exercises

15.7 Triple Integrals in Cylindrical Coordinates

- Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ using cylindrical coordinates

Fin

This concludes session 6.