Math 215 Mini Textbook

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Contents

0.1 Three Dimensional Coordinate Systems

In three-dimensional coordinate systems, points are represented by ordered triples (x, y, z). The three axes (x, y, and z) are mutually perpendicular, and the position of a point is determined by its distances from these axes.

0.2 Vectors

Vectors are mathematical objects that have both magnitude and direction. They are often represented as directed line segments or as ordered triples (x, y, z) in three-dimensional space. Vectors can be added together and multiplied by scalars.

0.3 Dot Product

The dot product (or scalar product) of two vectors is a way of multiplying them to get a scalar. For vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, the dot product is given by $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. It is used to find the angle between vectors and to determine orthogonality. The cosine of the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be found using the dot product and the magnitudes of the vectors. The formula is given by:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, respectively.

0.4 Cross Product

The cross product (or vector product) of two vectors in three-dimensional space results in a third vector that is perpendicular to the plane containing the original vectors. For vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product $\mathbf{a} \times \mathbf{b}$ is given by a determinant involving the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The cross product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Expanding the determinant, we get:

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

0.5 Equation of lines and planes

The equation of a line in three-dimensional space can be written in parametric form using a point and a direction vector. The equation of a plane can be written in the form $Ax + By + Cz = D$, where A, B, and C are the coefficients that define the normal vector to the plane.

0.6 Cylinders

Cylinders are surfaces generated by moving a line (the generator) parallel to itself along a curve (the directrix). In three-dimensional space, a common type of cylinder is the right circular cylinder, which has a circular base and a fixed height.

0.7 Quadric Surfaces

Quadric surfaces are the graphs of second-degree equations in three variables. Examples include ellipsoids, hyperboloids, paraboloids, and cones. These surfaces can be classified based on the signs and values of the coefficients in their defining equations.

0.8 Vector Functions and Space Curves

0.8.1 Vector Function Definition

A vector function is a function that takes a real number as input and outputs a vector. It can be written as $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $f(t)$, $g(t)$, and $h(t)$ are scalar functions of t.

0.8.2 Space Curve Definintion

A space curve is the set of all points $\mathbf{r}(t)$ in space as t varies over an interval. It can be thought of as the path traced out by a particle moving in space.

0.8.3 Integrals, Derivatives, and Limits

Consider a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

- The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.
- The integral of $\mathbf{r}(t)$ is $\int \mathbf{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$.
- The limit of $\mathbf{r}(t)$ as t approaches t_0 is $\lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \rangle$.

0.8.4 Tangent Lines

The tangent line to the curve $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$ is the line that passes through $\mathbf{r}(t_0)$ and has the same direction as the velocity vector $\mathbf{r}'(t_0)$.

0.9 Arc Length

Given a curve parameterized by $\mathbf{r}(t)$ the arc length L between $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is given by:

$$L = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt$$

0.10 Motion in Space

Given a position vector $\mathbf{r}(t)$ that describes the position of a particle at time t, the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$ and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

0.11 Functions of Several Variables

0.11.1 Function of Serveral Variables Definition

A function of several variables is a function that takes two or more variables as input and produces a single output. For example, a function f of two variables x and y can be written as $f(x, y)$. The domain of f is the set of all pairs (x, y) for which $f(x, y)$ is defined, and the range of f is the set of all possible values of $f(x, y)$.

0.11.2 Level Curves

A level curve is given by $k = f(x, y)$, where k is a constant. It represents the set of all points (x, y) in the domain of f where the function $f(x, y)$ takes on the same value k. Level curves are useful for visualizing functions of two variables, as they provide a way to see how the function behaves in different regions of its domain.

0.11.3 Contour Map

A contour map is a graphical representation of a function of two variables, $f(x, y)$, where contour lines are drawn to connect points that have the same function value. Each contour line represents a specific value of the function, and the spacing between the lines indicates the rate of change of the function. Contour maps are useful for visualizing the topography of a surface, as they provide a way to see how the function values change over the domain.

0.11.4 Contour Surfaces (Extending level curves to higher dimensions)

A contour surface is the three-dimensional analog of a contour line (or level curve). It is a surface in three-dimensional space representing points where a function of three variables $f(x, y, z)$ is constant. For example, the equation $f(x, y, z) = k$ defines a contour surface for a constant k. Contour surfaces are useful for visualizing functions of three variables, as they provide a way to see how the function behaves in different regions of its domain.

0.12 Partial Derivatives

0.12.1 Definition

Partial derivatives are the derivatives of functions of multiple variables with respect to one variable, while keeping the other variables constant. For a function

$f(x, y)$, the partial derivative with respect to x is denoted by $\frac{\partial f}{\partial x}$ and is defined as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly, the partial derivative with respect to y is denoted by $\frac{\partial f}{\partial y}$ and is defined as:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivatives are used to analyze the rate of change of a function with respect to each of its variables independently.

0.12.2 Theorem

If f_{xy} and f_{yx} are continuous then we have $f_{xy} = f_{yx}$

0.13 Tangent Planes and Linear Approximations

0.13.1 Tangent Planes

Given a differentiable function $f(x, y)$, the equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

where f_x and f_y are the partial derivatives of f with respect to x and y, respectively, evaluated at (x_0, y_0).

0.13.2 Linear Approximations

The linear approximation (or tangent plane approximation) of a function $f(x, y)$ near a point (x_0, y_0) is given by:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This approximation is useful for estimating the value of the function near the point (x_0, y_0) using the values of the function and its partial derivatives at that point.

0.14 Maximum and Minimum Values: Local Extrema

0.14.1 Second Derivative Test

$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

(1) If $H > 0$ and $f_{xx} > 0$, then f has a local minimum at (a, b).

(2) If $H > 0$ and $f_{xx} < 0$, then f has a local maximum at (a, b).

(3) If $H < 0$, then f has a saddle point at (a, b).

(4) If $H = 0$, then the test is inconclusive.

0.15 Maximum and Minimum Values: Global Extrema

0.15.1 Extreme Value Theorem

If f is continuous on a closed and bounded set D, then f has both a maximum and minimum value on D.

(1) Evaluate f at all critical points in D.

(2) Find the maximum and minimum values of f on the boundary of D.

(3) Compare the values from steps (1) and (2) to find the global maximum and minimum values of f on D.

(4) Largest value is the global maximum, smallest value is the global minimum.

0.16 Lagrange Multipliers

To find the maximum and minimum values of a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, we solve the system of equations:

(1) $\nabla f = \lambda \nabla g$

(2) $g(x, y, z) = k$

(3) where $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$ and $\nabla g = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$

(4) and λ is the Lagrange multiplier.

(5) The solutions to the system of equations are the critical points of f subject to the constraint $g(x, y, z) = k$.

0.17 Double Integrals over Rectangles

For a function $f(x, y)$ defined on a rectangle $R = [a, b] \times [c, d]$, the double integral of f over R is defined as

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta x \Delta y$$

0.17.1 Fubini's Theorem

If $f(x, y)$ is continuous on a rectangle $R = [a, b] \times [c, d]$, then

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

0.18 Double Integrals over General Regions

Let $f(x, y)$ be a continuous function defined on a region D in the xy-plane.

0.18.1 Type I Regions

A region D is a **Type I** region if it is bounded by the graphs of two functions $y = g_1(x)$ and $y = g_2(x)$, and the lines $x = a$ and $x = b$. The double integral of f over D is

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

0.18.2 Type II Regions

A region D is a **Type II** region if it is bounded by the graphs of two functions $x = h_1(y)$ and $x = h_2(y)$, and the lines $y = c$ and $y = d$. The double integral of f over D is

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

0.18.3 Properties

- The area of D is given by

$$\text{Area}(D) = \iint_D 1\, dA$$

- The average value of f over D is given by

$$\text{Avg}(f) = \frac{1}{\text{Area}(D)} \iint_D f(x, y)\, dA$$

- The net volume of the solid bounded by the surface $z = f(x, y)$ and the region D is given by

$$\text{Volume} = \iint_D f(x, y)\, dA$$

- Pay attention to symmetry when setting up double integrals over general regions. If D is symmetric with respect to the x-axis, y-axis, or origin, you can take advantage of this symmetry to simplify the integral.

0.19 Polar Coordinates

0.19.1 Converting Between Rectangular and Polar Coordinates

The conversion formulas between rectangular and polar coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

0.19.2 Double Integrals in Polar Coordinates

If $f(x,y)$ is a continuous function defined on a region D in the xy-plane, then the double integral of f over D can be expressed in polar coordinates as

$$\iint_D f(x,y)\,dA = \iint_E f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where D is the region in the xy-plane that corresponds to the region E in the $r\theta$-plane, and $dA = r\,dr\,d\theta$.

0.20 Applications of Double Integrals

0.20.1 Mass

Consider a lamina which occupies a region D on the xy-plane. The density of the lamina at a point (x,y) is given by $\delta(x,y)$. The mass of the lamina is given by the double integral

$$\int\int_D \delta(x,y)dA$$

where $dA = dxdy$.

0.20.2 Center of Mass

The center of mass of the lamina is given by the point $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{1}{M}\int\int_D x\delta(x,y)dA$$

$$\bar{y} = \frac{1}{M}\int\int_D y\delta(x,y)dA$$

and M is the mass of the lamina.

0.20.3 Surface Area

The surface area of a surface S on domain D is given by the double integral

$$\int\int_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dA$$

where $z=f(x,y)$ is the equation of the surface.

0.21 Triple Integrals

Let f(x,y,z) be a function defined on a volume V in $\mathbb{R}^3$. The triple integral of f over V is given by

$$\int\int\int_V f(x,y,z)dV$$

where $dV=dxdydz$. If V is defined by $a\leq x\leq b$, $c\leq y\leq d$, and $e\leq z\leq f$, then the triple integral can be written as

$$\int_c^d\int_a^b\int_e^f f(x,y,z)dzdxdy$$

0.21.1 Applications

- **Volume:** The volume of a solid occupying a region V in $\mathbb{R}^3$ is given by

$$\int\int\int_V dV$$

- **Mass:** The mass of a solid occupying a region V in $\mathbb{R}^3$ with density $\delta(x,y,z)$ is given by

$$\int\int\int_V \delta(x,y,z)dV$$

- **Center of Mass:** The center of mass of a solid occupying a region V in $\mathbb{R}^3$ is given by the point $(\bar{x},\bar{y},\bar{z})$ where

$$\bar{x}=\frac{1}{M}\int\int\int_V x\delta(x,y,z)dV$$

$$\bar{y}=\frac{1}{M}\int\int\int_V y\delta(x,y,z)dV$$

$$\bar{z}=\frac{1}{M}\int\int\int_V z\delta(x,y,z)dV$$

and M is the mass of the solid.

0.22 Triple Integrals in Cylindrical Coordinates

In cylindrical coordinates, the triple integral of a function $f(x, y, z)$ over a volume V is given by

$$\int\int\int_V f(x,y,z)dV = \int\int\int_V f(r\cos(\theta), r\sin(\theta), z)rdzdrd\theta$$

where $dV = rdzdrd\theta$.

0.23 Vector Fields

A vector field is a function that assigns a vector to each point in space. A vector field $\mathbf{F}$ in $\mathbb{R}^3$ can be represented as:

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

where P, Q, and R are scalar functions of x, y, and z.

(i) $\mathbf{F}$ is continuous if and only if coordinate functions P, Q, and R are continuous.

(ii) $\mathbf{F}$ is differentiable if and only if coordinate functions P, Q, and R are differentiable.

0.24 Line Integrals

Let C be a smooth curve parameterized by $(r(t))$ on $[a, b]$.

(i) The line integral of a scalar function f along the curve C is given by:

$$\int_C f(x,y,z)ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$$

where $ds = \|\mathbf{r}'(t)\|dt$ is the differential arc length.

(ii) The line integral of a vector field $\mathbf{F}$ along the curve C is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

where $d\mathbf{r} = \mathbf{r}'(t)dt$ is the differential vector along the curve. Note that if $\mathbf{F}$ is a force field, then the line integral represents the work done by the force field along the curve C.

0.25 Fundamental Theorem for Line Integrals

Let C be a smooth curve parameterized by $(r(t))$ on $[a, b]$. If $\mathbf{F}$ is a conservative vector field, then there exists a scalar potential function f such that $\nabla f = \mathbf{F}$. The fundamental theorem for line integrals states that:

(i) The line integral of $\mathbf{F}$ along the curve C is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where f is the potential function for $\mathbf{F}$.

(ii) The line integral is independent of the path taken from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. And if C is a closed curve, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

0.25.1 When is F conservative?

A vector field $\mathbf{F}$ is conservative if and only if the following conditions are satisfied:

(i) The vector field is defined on a simply connected domain.

(ii) The curl of the vector field is zero:

$$\nabla \times \mathbf{F} = 0$$

In two dimensions this is to say that

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

0.26 Green's Theorem

Theorem 1 (Green's Theorem)**.** *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ are functions of (x,y) with continuous partial derivatives on an open region that contains D, then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

In this theorem, P and Q are the components of a vector field $\mathbf{F} = (P, Q)$, and the left-hand side is the line integral of $\mathbf{F}$ around the curve C. The right-hand side is the double integral of the curl of $\mathbf{F}$ over the region D.

0.27 Curl and Divergence

0.27.1 Curl

Theorem 2 (Curl)**.** *Let $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field in $\mathbb{R}^3$. The curl of $\mathbf{F}$ is defined as*

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

The curl of a vector field measures the rotation or "twisting" of the field at a point. It is a vector quantity that points in the direction of the axis of rotation, and its magnitude represents the strength of the rotation.

Conservative Fields. If the curl of a vector field $\mathbf{F}$ is zero, i.e., $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is said to be a conservative vector field. This means that $\mathbf{F}$ can be expressed as the gradient of a scalar potential function ϕ, such that $\mathbf{F} = \nabla\phi$. In physical terms, this implies that the work done by the field along any closed path is zero, and the field is path-independent.

0.27.2 Divergence

Theorem 3 (Divergence)**.** *Let $\mathbf{F} = (P(x, y, z), Q(x, y, z), R(x, y, z))$ be a vector field in $\mathbb{R}^3$. The divergence of $\mathbf{F}$ is defined as*

$$div\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a vector field measures the "spreading out" or "convergence" of the field at a point. It is a scalar quantity that indicates whether the field is diverging from or converging towards that point.

Propisition. The divergence of the curl of any vector field is zero, i.e., $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. This means that the curl of a vector field has no net "outflow" at any point in space. This is a consequence of the fact that the curl measures rotation, while divergence measures "spreading out".

0.28 Parametric Surfaces and their Areas

Definition 1. *A parametric surface is a surface in $\mathbb{R}^3$ defined by a vector-valued function $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, where (u, v) are parameters that vary over some region in the uv-plane. The surface is the image of this parameterization.*

Consider the vector function $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$

- The partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v are given by:

  $$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

  These vectors lie in the tangent plane to the surface at a given point.

- The normal vector to the tangent plane at a point on the surface is given by the cross product of the partial derivatives:

  $$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}.$$

  This vector is perpendicular to the tangent plane and can be used to compute the surface area.

0.29 Surface Integrals (Scalar Functions)

Definition 2. *Given a surface S parameterized by $\mathbf{r}(u, v)$, the surface integral of a scalar function $f(x, y, z)$ over the surface S is defined as:*

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv$$

where D is the parameter domain in the uv-plane, and $\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|$ is the magnitude of the cross product of the partial derivatives, which gives the area element of the surface.

Note that to compute the area of the surface integral S, we can set $f(x, y, z) = 1$. The surface integral then becomes the area of the surface:

$$\iint_S dS = \iint_D \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv$$

0.30 Surface Integrals (Vector Fields)

Definition 3. *Given a surface S parameterized by $\mathbf{r}(u, v)$ with a unit normal vector $\mathbf{n}$, the surface integral of a vector field $\mathbf{F}(x, y, z)$ over the surface S is defined as:*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv$$

where D is the parameter domain in the uv-plane, and $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is the cross product of the partial derivatives, which gives the oriented area element of the surface.

- If $\mathbf{F}$ is a velocity field of a fluid, then the flux

  $$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$$

  is the rate of flow across the surface S.

- The orientation of S is determined by the choice of the unit normal vector $\mathbf{n}$. The direction of $\mathbf{n}$ can be chosen to be outward or inward, depending on the context of the problem.

0.31 Stokes' Theorem

Theorem 4 (Stokes' Theorem)**.** *Let S be a smooth, oriented surface with boundary curve C, and let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. Then,*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $d\mathbf{S} = \mathbf{n}\, dS$ is the oriented area element of the surface S, and $d\mathbf{r}$ is the line element along the curve C.

If $S = D$ is a domain in $\mathbb{R}^2$ and $C = \partial D$ is the boundary of D, then Stokes' theorem reduces to Green's theorem.

0.32 Divergence Theorem

Theorem 5 (Divergence Theorem)**.** *Let V be a solid region in $\mathbb{R}^3$ with a piecewise smooth boundary surface S, oriented outward. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing V. Then,*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

where $\mathbf{n}$ is the outward unit normal vector to S, and $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$.

- The divergence theorem relates the flux of a vector field through a closed surface S to the triple integral of the divergence of the field over the volume V enclosed by S.
- The surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ represents the total flux of the vector field $\mathbf{F}$ across the boundary surface S.
- The volume integral $\iiint_V (\nabla \cdot \mathbf{F})\, dV$ represents the total "outflow" of the vector field $\mathbf{F}$ from the volume V.
- The divergence theorem is also known as Gauss's theorem or Ostrogradsky's theorem.
- The theorem requires that the vector field $\mathbf{F}$ is continuously differentiable and that the region V is bounded and has a well-defined, piecewise smooth boundary.